NEW ANALYSES OF DISTRIBUTED RC INTERCONNECTIONS

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ABSTRACT

This paper gives new methods for calculating the time-domain response for a finite-length distributed RC line. We begin with the solution for diffusion in a semi-infinite distributed RC line [4, 12] and apply the method of images (reflections) to obtain the voltage response for a *finite-length* RC line. Second, we describe a general approach for calculating the time-domain response for finite RC lines with source and load impedances. Next, we obtain new approximate analytical expressions for the voltage response in a finite RC line with resistive source and capacitive load. Delay estimates using this method are very close to SPICE-computed delays; we compare against both SPICE and the recent method of [12] which calculates delays in an RC line with capacitive load. Our method is simple; for higher accuracy in the response, additional terms can be computed using numerical techniques.

1. INTRODUCTION

Estimating delays on VLSI interconnections is a key element of timing verification, gate-level simulation and performance-driven layout design. Because of the highly resistive nature of the wires, many tools model the wires inside current integrated circuits as distributed RC lines. The open-ended finite RC transmission line is widely analyzed in, e.g., [11, 14, 13]. The standard approach is to first calculate the transfer function; then, by approximating the transfer function both transform-domain and time-domain responses are obtained for such simple cases as a short-circuit or perfectly matched load [11, 14, 1, 10]. Using a different approach to invert the Laplace transform of the response, [9, 5, 12] have all obtained the exact time-domain response for a finite-length open-ended RC line. The most recent of these works, by Rao [12], extends the traditional transform-domain analysis to calculate an approximate time-domain response for a finite RC line with capacitive load impedance. The time-domain response is approximated by calculating 10 dominant poles and their corresponding residues from the transform-domain response; pole values are calculated by solving the transcendental equation numerically.

A direct solution of the open-ended finite RC line response, i.e., directly in the time-domain, was first given by Kaufman and Garrett [7]. To obtain the transient response to a step input, [7] makes the simplifying assumption $v(x,t) = f(x)\cdot g(t)$, i.e., that the voltage response is separable into functions of position and time. An infinite series is obtained, and the response can be approximated by considering as many dominant poles as needed. More recently, Kahng and Muddu [4] also gave a direct time-domain analysis of voltage response: in a finite distributed RC line, the total response was shown to be equal to the infinite sum of diffusion equation solutions, with each diffusion starting at either end of the line. Various cases of source and load impedances either infinity or zero were discussed, e.g., the open-ended line, short-circuit load, and perfectly matched load. Analysis of the open-ended line case in [4] was incorrect; [12] has pointed out that the response given is actually that of the semi-infinite line.

In this paper, we make the following contributions.

1. We begin with the solution of diffusion in a semi-infinite distributed RC line [4, 12] and apply the method of images (reflections) to obtain the correct voltage response for a finite RC line. The total response is obtained by adding various individual diffusion components; each of these can be viewed as traveling either forward or backward along the line, in analogy with reflections.
2. We obtain new approximate analytical expressions for the voltage response in a finite RC line with capacitive load and resistive source impedance. Delay estimates are very close to SPICE-computed delays; we give comparisons with both SPICE and, for the case of only a capacitive load, the recent method of [12]. The time complexity of invoking these analytical expressions is essentially a constant. For higher accuracy, additional terms can be obtained using numerical integration. Our analysis of RC lines extends to response computations in general interconnection trees by considering both reflection and transmission components at discontinuities.

2. DIFFUSION-BASED RC LINE ANALYSIS

2.1. Semi-Infinite Line

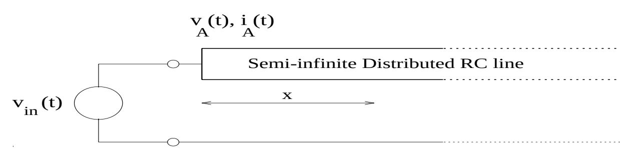

Figure 1. A semi-infinite distributed RC line, and position x along the line.

Consider the semi-infinite distributed RC line shown in Figure 1. The governing PDEs are

$$\frac{\partial i(x,t)}{\partial x} = -c(x)\frac{\partial v(x,t)}{\partial t} \qquad \frac{\partial v(x,t)}{\partial x} = -r(x)i(x,t)$$

where $r(x)$ and $c(x)$ are resistance and capacitance per unit length. In a uniform wire, $r(x) = r$ and $c(x) = c$ are constants and we obtain the diffusion equation

$$rc\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \ .$$

Using the substitution $\eta = x\sqrt{\frac{rc}{2t}}$, the general solution is [4]

$$v(\eta) = C_1 \int_0^{\eta} e^{-\frac{\eta^2}{2}} d\eta + C_2 \ . \qquad (1)$$

Initial and boundary conditions for the semi-infinite line are

$$\begin{array}{llcll} \text{IC:} & v(x,0) & = & 0 & \text{for all } x \geq 0 \\ \text{BC1:} & v(0,t) & = & v_A(t) & \text{for all } t \geq 0 \end{array} \qquad (2)$$

and the initial condition that the line is quiet at $t = 0$ implies $C_1 = -\sqrt{\frac{2}{\pi}}C_2$. For ideal source and step input, the response is

$$v_I(x,t) = V_0 \ erfc(x\sqrt{\frac{rc}{4t}}) = V_0 \ erfc(\sqrt{\frac{R_x C_x}{4t}}) \qquad (3)$$

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where $R_x = xr$, and $C_x = xc$. Later, we will use the transform domain analog of Equation (3), $V_I(x,s) = \frac{V_0}{s} \cdot e^{-x\sqrt{rcs}}$; also, for a general input at the source end of the line $V_A(s)$ the transform domain response is $V_I(x,s) = V_A(s) \cdot e^{-x\sqrt{rcs}}$.

2.2. Finite Distributed RC Line

An actual distributed transmission line has finite length, implying a change of impedance at the end of the line. For typical VLSI interconnections, discontinuities at the end of the line are due to capacitive load, and those at the source are due to a resistive driver. These impedance changes cause the voltage on the line to evolve according to proper reflections. The reflection coefficient at the discontinuity between two lines is $\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$, and the transmission coefficient is $(1 + \Gamma)$, where Z_1 and Z_2 are the respective characteristic impedances of the lines. The concepts of reflected and transmitted voltages apply at any discontinuity; we use reflection coefficients to calculate voltage propagation in a finite distributed RC line.

In a finite RC line (Figure 3), the diffusion equation solution of Equation (3) can be viewed as an *incident* propagation of voltage due to a step input. The Laplace transform of this incident voltage propagation [3] is $V_I(x,s) = \frac{V_0}{s} e^{-\sqrt{R_x C_x s}}$. The total voltage on the line is the summation of the incident diffusion component plus *reflected diffusion components* due to discontinuities at the source (S) and load (L). In other words, the time-domain expression for total voltage is $v_{Tot}(x,t) = v_I(x,t) + \sum_{i=1}^{\infty} v_{R_i}(x,t)$, where $v_I(x,t) \equiv$ voltage due to the incident diffusion and $v_{R_i}(x,t) \equiv$ voltage due to the i^{th} reflection.[1] The expression for any $v_{R_i}(x,t)$ will be of the same form as the expression for the incident voltage $v_I(x,t)$, but with a displacement in the position variable and with a different magnitude (according to the reflection coefficients).

In general, $v_{R_i}(x,t)$ can be calculated either through convolution of the reflected diffusion taking into account position displacement, with the reflection coefficients or through inverse Laplace transform of $V_{R_i}(x,s)$. For example, the voltage at position x (Figure 3) due to the first reflection at the load can be calculated by considering the incident wave and shifting in position by $h + h - x = 2h - x$ to obtain $V_{R_1}(x,s) = \Gamma_L(s) V_I(2h - x, s)$, i.e., the first reflected voltage travels a distance h to the end of the line before reflection and then $h - x$ to reach the desired location.[2] The reflection voltages are also explained by applying the symmetry argument in the Method of Images (or Reflections) [8] to satisfy the boundary condition at the end of the line $x = h$ (Figure 2).
Similarly, the second reflection at the source yields $V_{R_2}(x,s) = \Gamma_S(s)\Gamma_L(s) V_I(2h + x, s)$, and in general the i^{th} reflection gives

$$V_{R_i}(x,s) = \begin{cases} \Gamma_L^{\frac{i}{2}}(s)\,\Gamma_S^{\frac{i}{2}}(s) V_I(ih + x) & \text{for } i \text{ even} \\ \Gamma_L^{\frac{i+1}{2}}(s)\,\Gamma_S^{\frac{i-1}{2}} V_I(h(i+1) - x) & \text{for } i \text{ odd} \end{cases}$$

so that

$$V_{Tot}(x,s) = V_I(x,s) + \sum_{n=1}^{\infty} \Gamma_L^n(s)\,\Gamma_S^{n-1} V_I(2nh - x, s) + \sum_{n=1}^{\infty} \Gamma_L^n(s)\,\Gamma_S^n(s) V_I(2nh + x, s)$$

[1] In our notation, R_i refers to the i^{th} *reflected diffusion* starting from either the source or the load discontinuity; i basically represents the number of trips up and down the line.

[2] The analysis of [4] neglected to shift in position and thus incorrectly obtained cancellations of reflected components in the response computation for the open-ended line case.

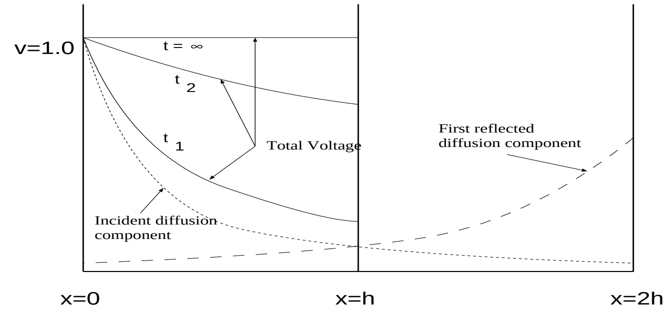

Figure 2. Evolution in time of the total voltage profile for a finite distributed RC line of length h. The small dotted line indicates the incident diffusion component and the large dotted line indicates the first reflection component. The total voltage on the line is equal to the sum of various diffusion componets.

with time-domain response

$$v_{Tot}(x,t) = v_I(x,t) + \sum_{n=1}^{\infty} \int_{\tau=0}^{t} a_n(t-\tau) v_I(2nh - x, \tau)\, d\tau + \sum_{n=1}^{\infty} \int_{\tau=0}^{t} b_n(t-\tau) v_I(2nh + x, \tau)\, d\tau \qquad (4)$$

where $a_n(t)$ and $b_n(t)$ represent odd and even n^{th} reflection coefficient values.

Our technical report [6] shows that both the diffusion equation approach described above and the 2-port transform domain approach can be made to yield the same result. For example, the finite open-ended RC line with ideal source has coefficients $\Gamma_S(s) = -1$ and $\Gamma_L(s) = 1$, and the time-domain expression for total voltage

$$v_{Tot}(x,t) = V_0 erfc(\sqrt{\frac{R_x C_x}{4t}}) + V_0 \sum_{n=1}^{\infty} (-1)^n erfc(\sqrt{\frac{R_{2nh+x} C_{2nh+x}}{4t}}) + V_0 \sum_{n=1}^{\infty} (-1)^{n-1} erfc(\sqrt{\frac{R_{2nh-x} C_{2nh-x}}{4t}}) \qquad (5)$$

has also been derived from the transform-domain approach in [5, 12]. However, even though the 2-port methodology has been widely used [14, 9, 12], the diffusion equation approach affords the insight to express the total voltage as a sum of reflected diffusion components. (The idea of reflection components is usually applied for distributed RLC lines which obey the wave equation for voltage propagation.) The remaining discussion demonstrates diffusion-based response computations for the cases of a line with capacitive load, and a line with resistive source and capacitive load.

3. FINITE RC LINE WITH CAPACITIVE LOAD

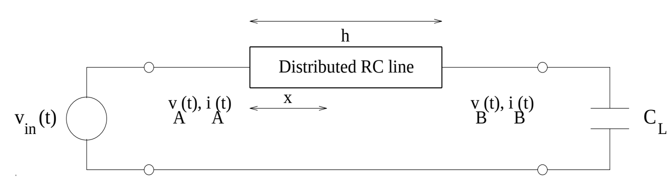

Figure 3. Distributed RC line of length h with capacitive load C_L.

We now consider a finite distributed RC line of length h with capacitive load C_L at the end of the line (Figure 3). This case has been previously treated in [12] via an extension of the classical transform-based analysis; the time-domain response was calculated by considering the first 10 dominant poles. From Equation (4), the total response is given by the infinite sum of all voltage components due to reflections at the load and source. It turns out that approximating the total response by summing only up to the first four reflections is quite close to the SPICE-computed response. If necessary (e.g., for larger loads than those we consider), more reflection components can be calculated using numerical techniques.

The following discussion will use the following shorthand quantities to express the time-domain voltage response in a concise form. Let

$$\begin{aligned} L1(y,n,m) &= (\frac{C_h}{y})^2 \frac{t}{R_h C_h} + \frac{(2nh+(-1)^m x)}{h} \frac{C_h}{y} \\ L2(y,n,m) &= \frac{C_h}{y} \sqrt{\frac{t}{R_h C_h}} + \frac{(2nh+(-1)^m x)}{2h} \sqrt{\frac{R_h C_h}{t}} \\ L3(y,n,m) &= (\frac{C_h}{y})^2 \frac{2t}{R_h C_h} + \frac{(2nh+(-1)^m x)}{h} \frac{C_h}{y} \qquad (6) \end{aligned}$$

where y can be either C_L or R_S, $n = 0, 1, 2, 3 \ldots$, and $m = 0$ for an even reflection component and $m = 1$ for an odd reflection component.

The reflection coefficient at the load for an impedance of $Z_L = \frac{1}{sC_L}$ is $, _L(s) = \frac{1-q\sqrt{s}}{1+q\sqrt{s}}$, where $q = \frac{C_L}{C_h}\sqrt{R_h C_h}$. The reflection coefficient at the source for an ideal source is $, _S(s) = -1$. The voltage response of the first reflected diffusion component at the load can be obtained as

$$V_{R_1}(x,s) = \frac{V_0}{s} \frac{1-q\sqrt{s}}{1+q\sqrt{s}} e^{-(2h-x)\sqrt{src}} = \frac{V_0}{s} \frac{1-q\sqrt{s}}{1+q\sqrt{s}} e^{-b\sqrt{s}}$$

where $b = \frac{(2h-x)}{h}\sqrt{R_h C_h}$. To compute the time-domain response we apply the following technique to invert the above Laplace transform response. Let

$$F_1(s) = V_0 \frac{(1-qs)}{s(1+qs)} e^{-bs} = V_0[\frac{1}{s} - \frac{2}{(s+1/q)}]e^{-bs}$$

Applying the inverse transform yields $f_1(t) = V_0(1 - 2e^{-\frac{(t-b)}{a}})U(t-b)$. The inverse transform of this function with $\sqrt{s}$ as transform variable can be obtained using the identity [6]

$$\frac{F_1(\sqrt{s})}{\sqrt{s}} \Longleftrightarrow \frac{1}{\sqrt{\pi t}} \int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} f_1(x) dx$$

Expressing the voltage response $V_{R_1}(x,s)$ in terms of the function F_1, we have $V_{R_1}(x,s) = \frac{F_1(\sqrt{s})}{\sqrt{s}}$ and corresponding time-domain response

$$\begin{aligned} v_{R_1}(x,t) &= \frac{1}{\sqrt{\pi t}} \int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} V_0(1 - 2e^{-\frac{(x-b)}{q}})U(x-b)dx \\ &= V_0 erfc\left(\frac{(2h-x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad -2V_0 e^{L1(C_L,1,1)} erfc(L2(C_L,1,1)) \end{aligned}$$

Similarly, the voltage response of the second reflected diffusion component (due to the source discontinuity with $, _S = -1$) is $V_{R_2}(x,s) = -, _L V_I(2h+x,s)$ and the corresponding time-domain response is

$$\begin{aligned} v_{R_2}(x,t) &= -V_0 erfc\left(\frac{(2h+x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad +2V_0 e^{L1(C_L,1,0)} erfc(L2(C_L,1,0)) \ . \end{aligned}$$

Extending, we may calculate the voltage response of the third reflected diffusion component as

$$V_{R_3}(x,s) = -\frac{V_0}{s}\left(\frac{1-q\sqrt{s}}{1+q\sqrt{s}}\right)^2 e^{-\frac{(4h-x)}{h}\sqrt{sR_h C_h}} = \frac{F_2(\sqrt{s})}{\sqrt{s}}$$

where $F_2(s) = -\frac{V_0}{s}(\frac{1-qs}{1+qs})^2 e^{-bs}$ and $b = \frac{(4h-x)}{h}\sqrt{R_h C_h}$. Substituting and again using the above identity [6], the time-domain expression for $v_{R_3}(x,t)$ is obtained as

$$\begin{aligned} v_{R_3}(x,t) &= \frac{-V_0}{\sqrt{\pi t}} \int_{x=b}^{\infty} e^{\frac{-x^2}{4t}} [1 - \frac{4(x-b)}{q} e^{-\frac{(x-b)}{q}}]dx \\ &= -V_0 erfc\left(\frac{4h-x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad + \frac{8V_0 C_h}{\sqrt{\pi} C_L}\sqrt{\frac{t}{R_h C_h}} e^{(L1(C_L,2,1) - L2(C_L,2,1)^2)} \\ &\quad -4V_0 L3(C_L,2,1) e^{L1(C_L,2,1)} erfc(L2(C_L,2,1)) \end{aligned}$$

Similarly, the voltage response of the fourth reflection at the source is $V_{R_4}(x,s) = , _L^2 V_I(4h+x,s)$ and the corresponding time-domain response is

$$\begin{aligned} v_{R_4}(x,t) &= V_0 erfc\left(\frac{4h+x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad - \frac{8V_0 C_h}{\sqrt{\pi} C_L}\sqrt{\frac{t}{R_h C_h}} e^{(L1(C_L,2,0) - L2(C_L,2,0)^2)} \\ &\quad +4V_0 L3(C_L,2,0) e^{L1(C_L,2,0)} erfc(L2(C_L,2,0)) \end{aligned}$$

The above procedure can be continued to obtain analytic expressions for the time-domain response for various diffusion

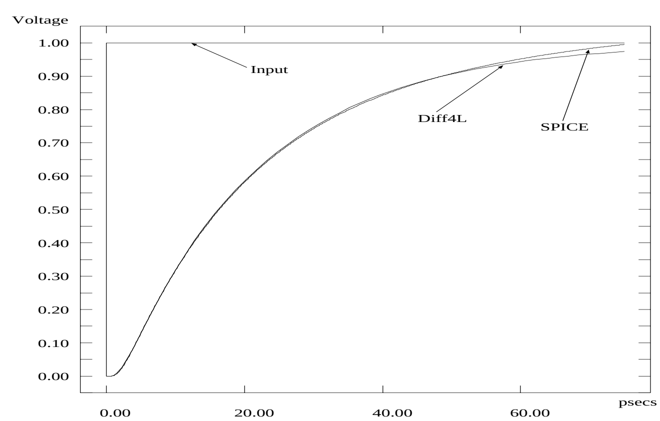

Figure 4. Voltage response at the end of finite RC line with load factor $\frac{C_L}{C_h} = 1.0$ using Diff4L model and SPICE. The line parameters are $r = 0.015\ \Omega/\mu m$, $c = 0.25\ fF/\mu m$ and length $h = 2000\mu m$. The time constant of the line is $R_h C_h = 15.0$ ps.

Load factor ($\frac{C_L}{C_h}$)	10% Threshold delays(R_hC_h)			50% Threshold delays (R_hC_h)		
	SPICE	**Rao95**	**Diff4L**	**SPICE**	**Rao95**	**Diff4L**
0.0	0.133	0.130	0.131	0.376	0.379	0.379
0.25	0.180	-	0.182	0.560	-	0.563
0.5	0.220	0.220	0.221	0.736	0.739	0.740
1.0	0.289	0.287	0.287	1.088	1.089	1.085
2.0	0.405	-	0.403	1.795	-	1.811
Load factor ($\frac{C_L}{C_h}$)	**63.2% Threshold delays(R_hC_h)**			**90% Threshold delays(R_hC_h)**		
	SPICE	**Rao95**	**Diff4L**	**SPICE**	**Rao95**	**Diff4L**
0.0	0.502	0.503	0.503	1.033	1.031	1.032
0.25	0.750	-	0.754	1.562	-	1.563
0.5	1.00	1.004	1.004	2.108	2.127	2.106
1.0	1.501	1.503	1.510	3.260	3.263	3.250
2.0	2.500	-	2.590	5.530	-	5.890

Table 1. Comparison of delay values at the end of the interconnect line ($x = h$) between SPICE, the method of [14] and the analytical Diff4L model which uses up to the first four reflected diffusion components.

components. In general, the diffusion components need not be evaluated analytically, but instead can be computed by numerical integration methods. Approximating the total response by considering only up to these first four reflections, we have

$$v_{Tot}(x,t) \approx v_I(x,t) + v_{R_1}(x,t) + v_{R_2}(x,t) + v_{R_3}(x,t) + v_{R_4}(x,t) \quad (7)$$

We call the approximation of Equation (7) the **Diff4L** model (i.e., up to 4 reflected diffusions, with capacitive Load impedance); Table 1 compares delay estimates at different threshold values for a wide range of capacitive loads, versus SPICE[3] and the method of [12]. Observe that delay estimates using our new diffusion equation approach are very close to the SPICE-computed delays (and to the method of [12] which is based on the first 10 poles of the response), even though only four reflections are considered. Figure 4 gives a comparison of voltage response between SPICE and **Diff4L** model for the case of load factor $\frac{C_L}{C_h} = 1.0$. Finally, we have also con-

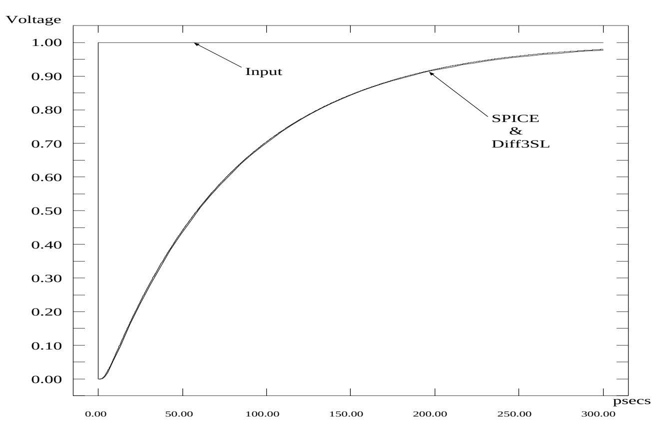

Figure 5. Voltage response at the end of finite RC line with source resistance $R_S = 2R_h$ and load capacitance $C_L = C_h$ using Diff3SL model and SPICE. The time constant of the line is $R_hC_h = 15.0$ psec.

sidered the finite-length distributed RC line with both finite resistive source impedance R_S and capacitive load C_L. Analytical expressions for incident and reflected diffusion components are given in [6]. Approximating the total voltage by considering only up to the first three reflected diffusion components yields our **Diff3SL** model (i.e., up to 3 reflected diffusions, with Source and capacitive Load impedance), which also matches the SPICE response very accurately. Figure 5 gives a comparison of voltage response between SPICE and **Diff3SL** model for the case of source resistance $R_S = 2R_h$ and load capacitance $C_L = C_h$.

[3] Note that SPICE uses the URC (Uniform Distributed RC) model for simulation of RC lines, i.e., it breaks each RC line into a finite number of RC segments.

4. CONCLUSIONS

In conclusion, we have proposed new methods which sum reflected diffusion components to approximate the time-domain response for a finite-length distributed RC line. The result obtained for the open-ended line is identical to that derived in the classical transform domain by [5, 12]. We have extended the technique for the more realistic cases of a finite RC line with capacitive load, and finite RC line with resistive source and capacitive load. The delay estimates using our new methods (incorporating only three or four reflected diffusion components) are very close to SPICE-computed delays. Our method is simple, and can achieve even higher accuracy in the response by computing additional terms using numerical techniques. Ongoing work with our new technique extends the analysis of RC lines to general interconnection trees by considering both reflection and transmission components.

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